Higher Derivatives

1. If $y = \left(x + \sqrt{x^2 + 1}\right)^p$, prove that $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - p^2y = 0$.

Differentiate this equation n times by Leibnitz's Theorem.

2. If $y = \frac{x}{1 + x^2}$, prove that $\frac{d^ny}{dx^n} = (-1)^n n! \cos(n+1)\theta \sin^{n+1}\theta$, where $x = \cot\theta$.

(Those who know complex numbers may try to prove without induction.)

3. If $y = \left(\sin^{-1}x\right)^2$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$.

Apply Leibnitz's theorem to this equation to find a relation between y_n, y_{n+1}, y_{n+2}, where $y_r = \frac{d^ry}{dx^r}$.

4. If $y = (x^2 - 1)^n$, prove that $(x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n + 1)y_n = 0$. If $P = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left\{(x^2 - 1)^n\right\}$, show that P

satisfies the equation: $(1 - x^2)y_2 + 2xy_1 - n(n + 1)y = 0$, where $y_r = \frac{d^ry}{dx^r}$.

5. If $y = (1 - x^2)^{1/2}\sin^{-1}x$, prove that

(i) $(1 - x^2)\frac{dy}{dx} + xy = 1 - x^2$ **(ii)** $(1 - x^2)\frac{d^{n+1}y}{dx^{n+1}} - (2n - 1)x\frac{d^ny}{dx^n} - n(n - 2)\frac{d^{n-1}y}{dx^{n-1}} = 0$, where $n > 2$.

6. Differentiate n times with respect to x, **(i)** $\frac{x}{x^2 - a^2}$ **(ii)** $\frac{x^2}{\left(x^2 - a^2\right)^2}$.

7. Prove that, if $y = x^2 \cos x$, then $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (x^2 + 6)y = 0$.

Deduce that, when $x = 0$, $(n - 2)(n - 3)\frac{d^ny}{dx^n} + n(n - 1)\frac{d^{n-2}y}{dx^{n-2}} = 0$

8. Show that if $y = (x - a)^n(x - b)^n$, $(x - a)(x - b)\frac{d^2y}{dx^2} - (n - 1)(2x - a - b)\frac{dy}{dx} - 2ny = 0$

and that, if $u = \frac{d^ny}{dx^n}$, $(x - a)(x - b)\frac{d^2u}{dx^2} + (2x - a - b)\frac{du}{dx} - n(n + 1)u = 0$.

9. If $y^2 = \sec 2x$, show that $\frac{d^2y}{dx^2} + y = 3y^5$.

10. If $x = c(2\cos\theta + \cos 2\theta)$, $y = c(2\sin\theta - \sin 2\theta)$, find $\frac{dy}{dx}$ in terms of θ.

and prove that : $8c\frac{d^2y}{dx^2} = \csc\frac{3}{2}\theta\sec^3\frac{1}{2}\theta$.

11. If $y = \lambda^m + \lambda^{-m}$, $x = \lambda + \lambda^{-1}$, prove that

(a) $(x^2 - 4)\left(\frac{dy}{dx}\right)^2 = m^2(y^2 - 4)$ **(b)** $(x^2 - 4)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y^2 = 0$.

12. If $x = \cos\theta$, $y = \cos^2 p\theta$, prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4p^2y - 2p^2 = 0$ and

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (4p^2 - n^2)\frac{d^n y}{dx^n} = 0 .$$

13. Show by Mathematical Induction that $\dfrac{d^n}{dx^n}\left(\dfrac{e^x}{x}\right) = (-1)^n n!\dfrac{e^x}{x^{n+1}}F_n(-x)$, where

$F_n(x) = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \ldots + \dfrac{x^n}{n!}$. Hence deduce that $\dfrac{d^{2n}}{dx^{2n}}\left(\dfrac{\sin x}{x}\right) = (-1)^n\dfrac{(2n)!}{x^{2n+1}}[C_{2n}(x)\sin x - S_{2n-1}(x)\cos x]$,

where $C_{2n}(x) = \displaystyle\sum_{i=0}^{n}(-1)^i\frac{x^{2i}}{(2i)!}$, $S_{2n-1}(x) = \displaystyle\sum_{i=0}^{n-1}(-1)^i\frac{x^{2i+1}}{(2i+1)!}$. (Complex number theory is needed.)

14. Show that the Tchebycheff polynomial : $T_m(x) = \dfrac{1}{2^{n-1}}\cos(m\cos^{-1}x)$ satisfies

$$(1-x^2)\,T_m''(x) - x\,T_m'(x) + m^2\,T_m(x) = 0 .$$

15. Show that the Legendre polynomial : $P_m(x) = \dfrac{1}{2^m m!}\dfrac{d^m}{dx^m}(x^2-1)^m$

$$(1-x^2)\,P_m''(x) - 2x\,P_m'(x) + m(m+1)\,P_m(x) = 0.$$

Hint : Differentiate $m + 1$ times the equation : $(x^2-1)\,u' = 2mxu$, where $u = (x^2-1)^m$.

16. If $y = \sin\ln(1+x)$, and $y_r = \dfrac{d^r y}{dx^r}$, prove that

(i) $(1+x)^2 y_2 + (1+x)y_1 + y = 0$ **(ii)** $(1+x)^2 y_{n+2} + (2n+1)(1+x)y_{n+1} + (n^2+1)y_n = 0$

17. If $y_0 = e^{x^2}$ and $y_r = \dfrac{d^r y_0}{dx^r}$ $\forall\, r\in\mathbf{N}$, prove that $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$.

If $u_n = e^{-x^2}y_n$, prove by induction with respect to r that, for $0 \le r \le n$,

$\dfrac{d^r u_n}{dx^r} = 2^r n(n-1)\ldots(n-r+1)u_{n-r}$, and hence evaluate $\dfrac{d^n u_n}{dx^n}$.

18. **(a)** Let $y = e^{ax}\sin bx$, where $a \ne 0$, $b \ne 0$. Prove that, for any positive integer n,

$\dfrac{d^n y}{dx^n} = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin(bx + n\phi)$, where $\cos\phi = \dfrac{a}{\sqrt{a^2+b^2}}$, $\sin\phi = \dfrac{b}{\sqrt{a^2+b^2}}$.

(b) For a function u , we define the n-th derivative $\dfrac{d^n u}{dx^n}$ by $u^{(n)}$ and $u^{(0)} = u$.

Prove that, for any positive integer n , $(u\cdot v)^{(n)} = \displaystyle\sum_{k=0}^{n}\binom{n}{k}u^{(n-k)}v^{(k)}$,

where $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$ and $0! = 1$.

(c) Making use of **(a)** and **(b)** , show that $(a^2+b^2)^{\frac{n}{2}}e^{ax}\sin(bx+n\phi) = \displaystyle\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\sin\left(bx + \frac{k\pi}{2}\right)$.